Spherical Geometry

Reference: Advanced level Pure Mathematics by S. L. Green Chapter 12 pp.189

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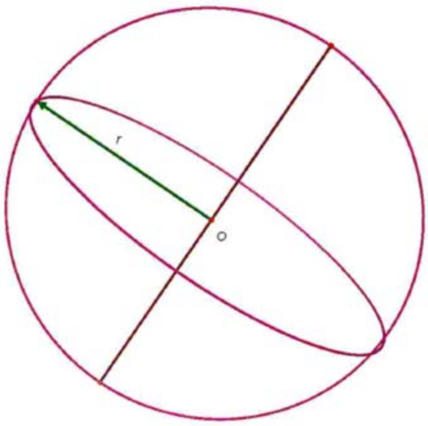
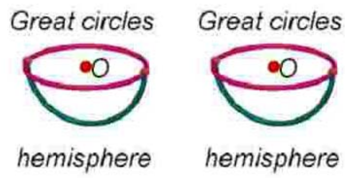
O is the **centre** of the sphere. Any point on the surface of the sphere is **equal distance** to the centre.
The distance is the **radius** *r* of the sphere.

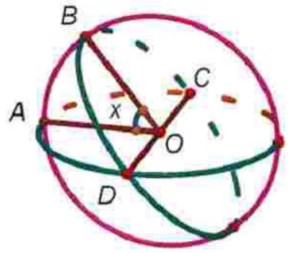
It is known that the surface area of a sphere is $4\pi r^2$.



A **great circle** is a circle on the surface of the sphere which cuts it into two equal hemi-spheres. The centre of a great circle is the centre *O* of the sphere.

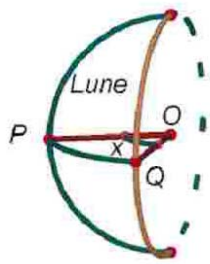


Suppose two great circles intersect at *C* and *D*. Then *CD* is the common diameter of the two great circles. Let *OA* be a radius on one great circle which is perpendicular to *CD*; and *OB* be a radius on the other great circle which is perpendicular to *CD*. The **angle** *x* between 2 great circles is the acute angle between the *OA* and *OB*; i.e. $\angle AOB$.



The **Lune** is a part of a surface of a sphere formed by two intersecting great circles. If the angle between the two great circles is $x°$, the surface area of a Lune is proportional to $x°$.

Area = $4\pi r^2 \times \frac{x^\circ}{360^\circ}$



A **triangle on the surface of a sphere** is a region formed by 3 great circles, where the 3 great circles do not intersect at a point at the same time.
The 3 great circles divide the surface of the sphere into 8 regions; namely *a*, *b*, *c*, *d*, *x*, *y*, *z*, *w*.

Note that $a = x$, $c = z$, $b = y$, $d = w$ ……………… (1)

$a + b + c + d + x + y + z + w$ = total surface area of the surface

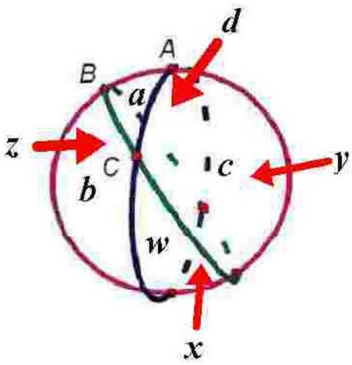
$a + b + c + d + x + y + z + w = 4\pi r^2$ ……………(2)

By (1), $a + b + c + d = x + y + z + w$ ……………(3)

$\therefore$ Sub. (3) into (2), $a + b + c + d = 2\pi r^2$ ………(4)

$a + b = 4\pi r^2 \times \frac{A}{360^\circ}$ ……………………………(5)

$a + c = 4\pi r^2 \times \frac{B}{360^\circ}$ ……………………………(6)



The red arrows show the regions behind the sphere, bounded by the dotted line.

$a + d = 4\pi r^2 \times \frac{C}{360^\circ}$ ……………………………(7)

(5) + (6) + (7): $2a + a + b + c + d = 4\pi r^2 \times \frac{A+B+C}{360^\circ}$ ………(8)

Sub. (4) into (8): $2a + 2\pi r^2 = 4\pi r^2 \times \frac{A+B+C}{360^\circ}$

$a = \frac{\pi r^2}{180^\circ} \times (A+B+C) - \pi r^2 = \frac{\pi r^2}{180^\circ} \times (A+B+C-180^\circ)$

Therefore, the area of ΔABC on the surface of a sphere is $\frac{\pi r^2}{180^\circ} \times (A+B+C-180^\circ)$ and the sum of angles of ΔABC is $A + B + C > 180°$

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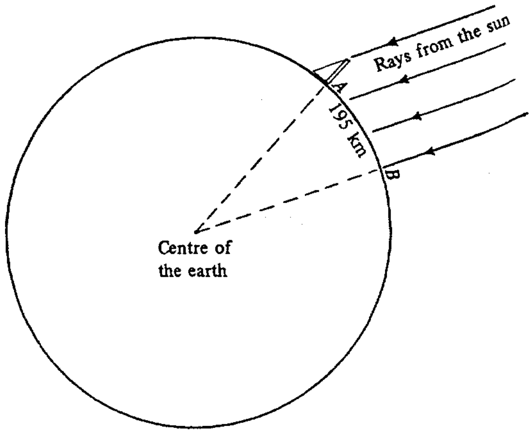
In the figure, town *A* is 195 km north of town *B*. At the moment when the sun is directly overhead in town *B*, a building 100 m tall at town *A* casts a shadow of 3 m on horizontal ground. With the above data, calculate the radius of the earth.



Rays from the sun can be taken to be parallel.

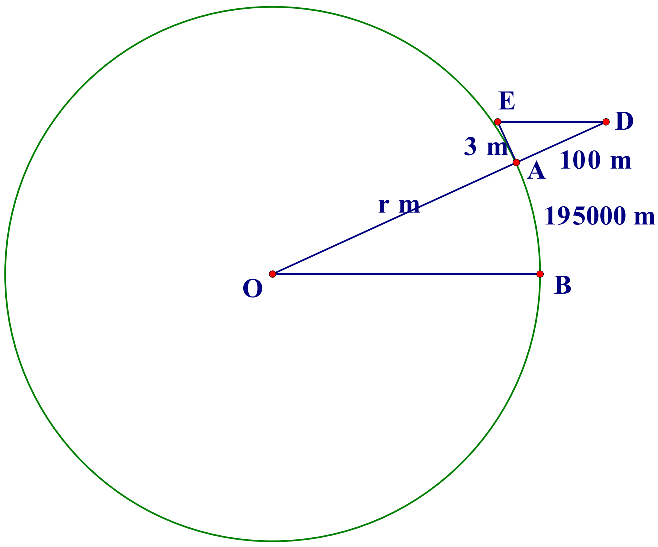
As shown in the figure, let *D* be the top of the tower, *E* be the tip of the shadow, the radius of the earth be *r* m.

Approximately, $\Delta ADE \sim \Delta AOB$.

$\frac{r}{195000} = \frac{100}{3}$ (ratio of sides, ~Δ's)

$r = 6500000$

$\therefore$ The radius of the earth is approximately 6500 km.



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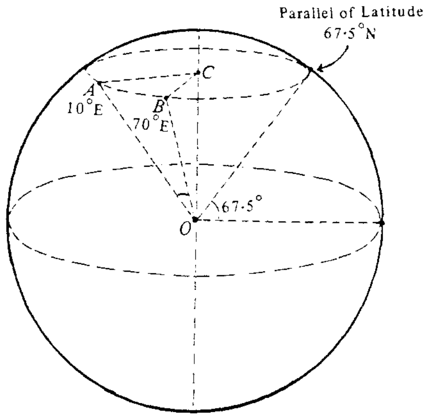

In the figure, *A* and *B* are two places on the earth's surface with the same latitude 67.5°N and with longitudes 10°E and 70°E respectively. *O* is the centre of the earth, and *C* the centre of the parallel of latitude 67.5°N.

(a) Calculate the distance from *A* to *B* along the latitude 67.5°N.

(b) Calculate $\angle AOB$.

(c) If a circle is described on the surface of the earth with centre *O*, passing through *A* and *B*, find the distance from *A* and *B* along the arc of this circle.

You may assume the earth to be a sphere of 6400 km.

(a) $\angle ACB = 70° - 10° = 60°$

$AC = BC = 6400 \times \cos 67.5°$ km

$\overset{\frown}{AB} = 2\pi AC \times \frac{60^\circ}{360^\circ} = 12800\pi \times \cos 67.5° \times \frac{60^\circ}{360^\circ}$ km

$= 2564.76$ km

(b) $AB = 2\,AC \sin 30° = AC = 6400 \times \cos 67.5°$ km

$\sin \frac{1}{2}\angle AOB = \frac{\frac{1}{2}AB}{OA} = \frac{3200 \times \cos 67.5^\circ}{6400}$

$\angle AOB = 22.062°$

(c) $\overset{\frown}{AB}$ along the great circle $= 2\pi \times 6400 \times \frac{22.062^\circ}{360^\circ}$ km

$= 2464.37$ km